ELEN E3401: Electromagnetics

Spring 2025

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Lecture #7

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

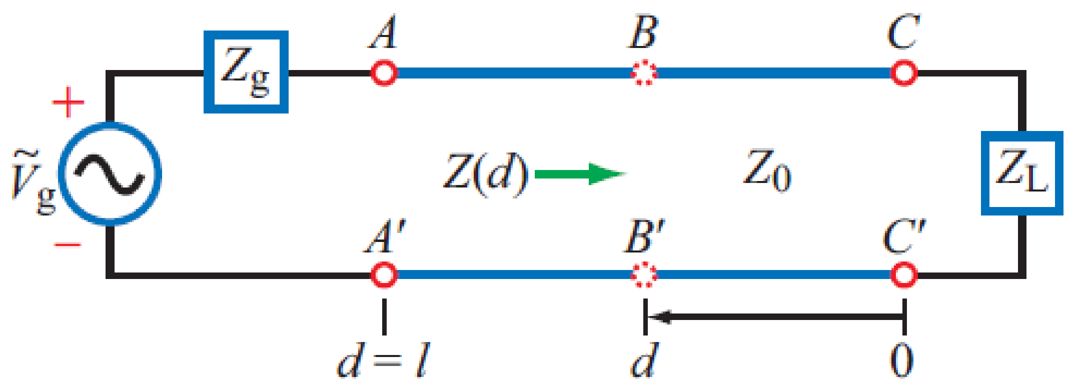
Wave Impedance – Lossless Line

When impedance not matched →
we have standing waves and voltage is out of phase with current.

We define the wave impedance:
ratio of the total voltage to the total current at any point, d, on the line.

Wave impedance:

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)}$$



Note: this is **not** $\mathbf{Z_0}$ which is the voltage/current ratio of individual wave amplitudes.

$$Z_0 = \frac{V_0^{\ +}}{I_0^{\ +}} = \frac{-V_0^{\ -}}{I_0^{\ -}}$$

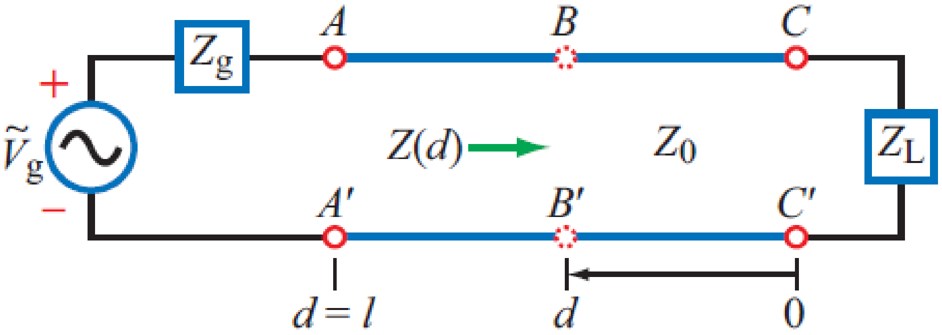
Wave Impedance – Lossless Line

Wave impedance: $Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)}$

Recall: $\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$ (wave equation solution)

Lossless → $\tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$

$$\tilde{V}(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z})$$

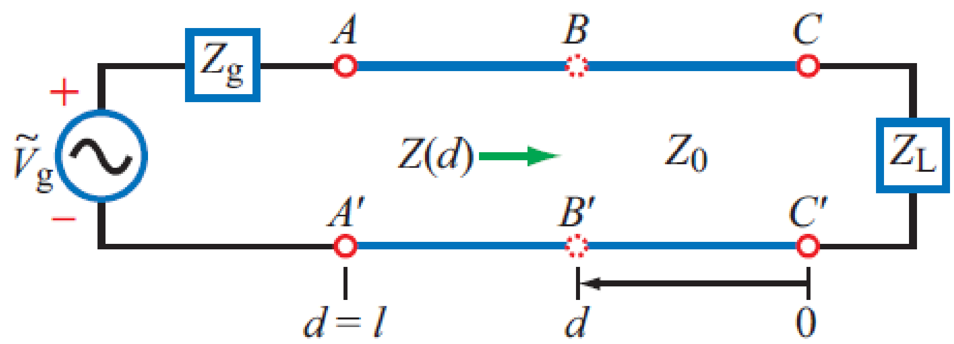


Then $z = -d$,

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+ (e^{j\beta d} + \Gamma e^{-j\beta d})}{V_0^+ (e^{j\beta d} - \Gamma e^{-j\beta d})} Z_0$$

Wave Impedance – Lossless Line

Wave impedance: $Z(d) = \dfrac{\tilde{V}(d)}{\tilde{I}(d)}$



$$Z(d) = Z_0\left(\frac{e^{j\beta d} + \Gamma e^{-j\beta d}}{e^{j\beta d} - \Gamma e^{-j\beta d}}\right) \times \frac{e^{-j\beta d}}{e^{-j\beta d}}$$

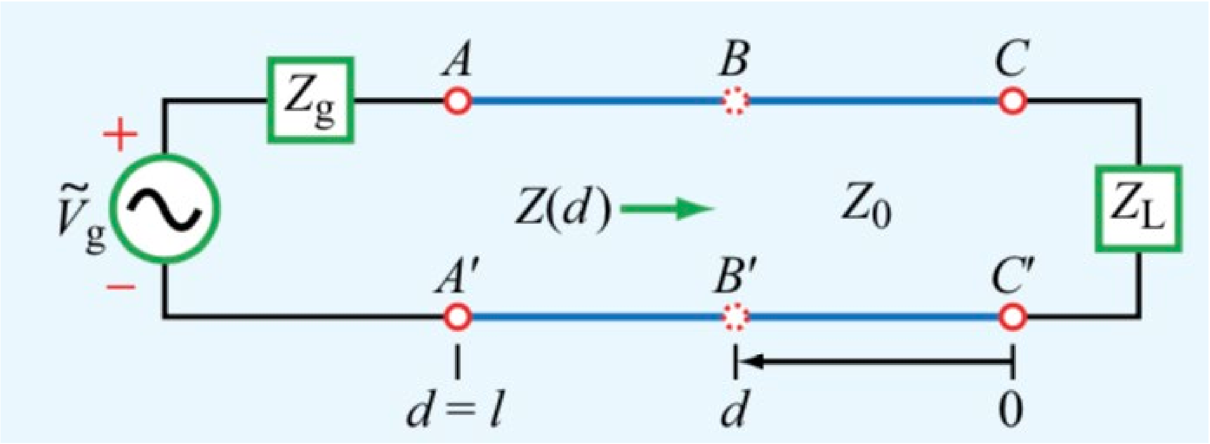
$$Z(d) = Z_0\left(\frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}}\right) = Z_0\left(\frac{1 + \Gamma_d}{1 - \Gamma_d}\right)$$

Define: $\Gamma_d = e^{-j2\beta d}\Gamma = |\Gamma| e^{j(\theta_r - 2\beta d)}$ → voltage reflection ratio, Γ, phase shifted by $2\beta d$

$(\Gamma_d = e^{-j2\beta d}\Gamma = |\Gamma| e^{j\theta_r - j2\beta d})$

Wave Impedance – Lossless Line

Consider the transmission line circuit:



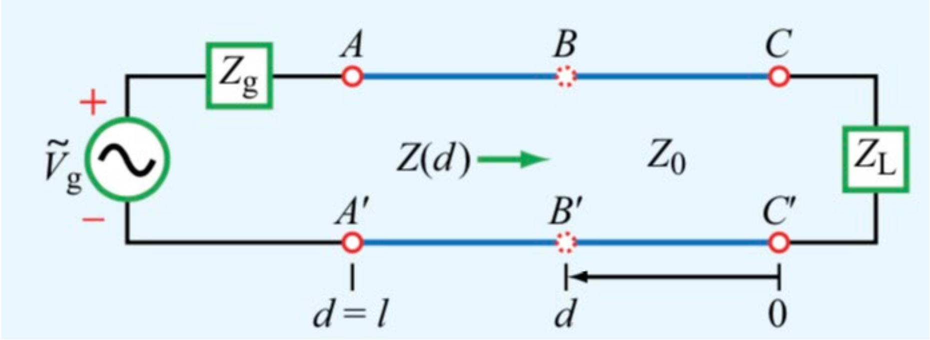
Replace with lumped impedance

Terminals BB' are at arbitrary location d on the line, Z(d):

wave impedance of line when "looking" toward load

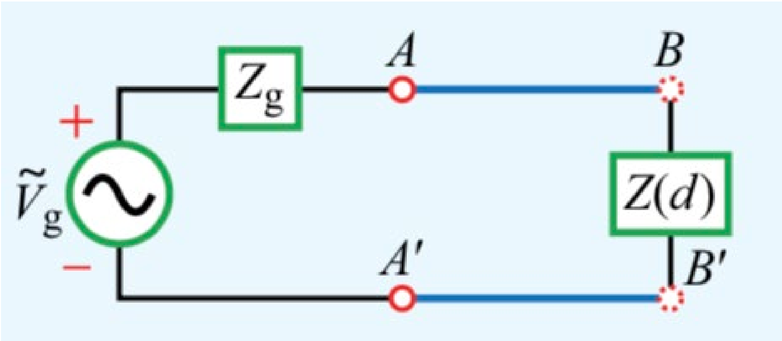
Wave Impedance – Lossless Line

Consider the transmission line circuit:



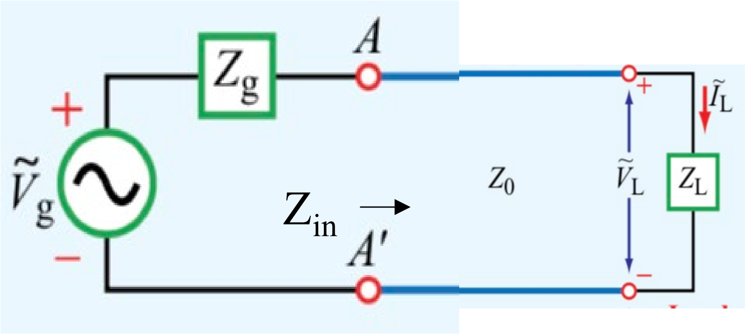
Replace with lumped impedance

We use equivalent circuit:
Replace segment to right of BB’ with lumped impedance → *Z(d)*



Lumped impedance → equivalent

Wave Impedance – Lossless Line



An important value is the input impedance at the source end of line, at $d = l$

$$Z_{in} = Z(l) = Z_0\left(\frac{1+\Gamma_l}{1-\Gamma_l}\right)$$

$$\Gamma = \frac{V_0^{\,-}}{V_0^{\,+}} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1}$$

Define: $\Gamma_l = e^{-j2\beta l}\Gamma = |\Gamma| e^{j\theta_r - j2\beta l}$

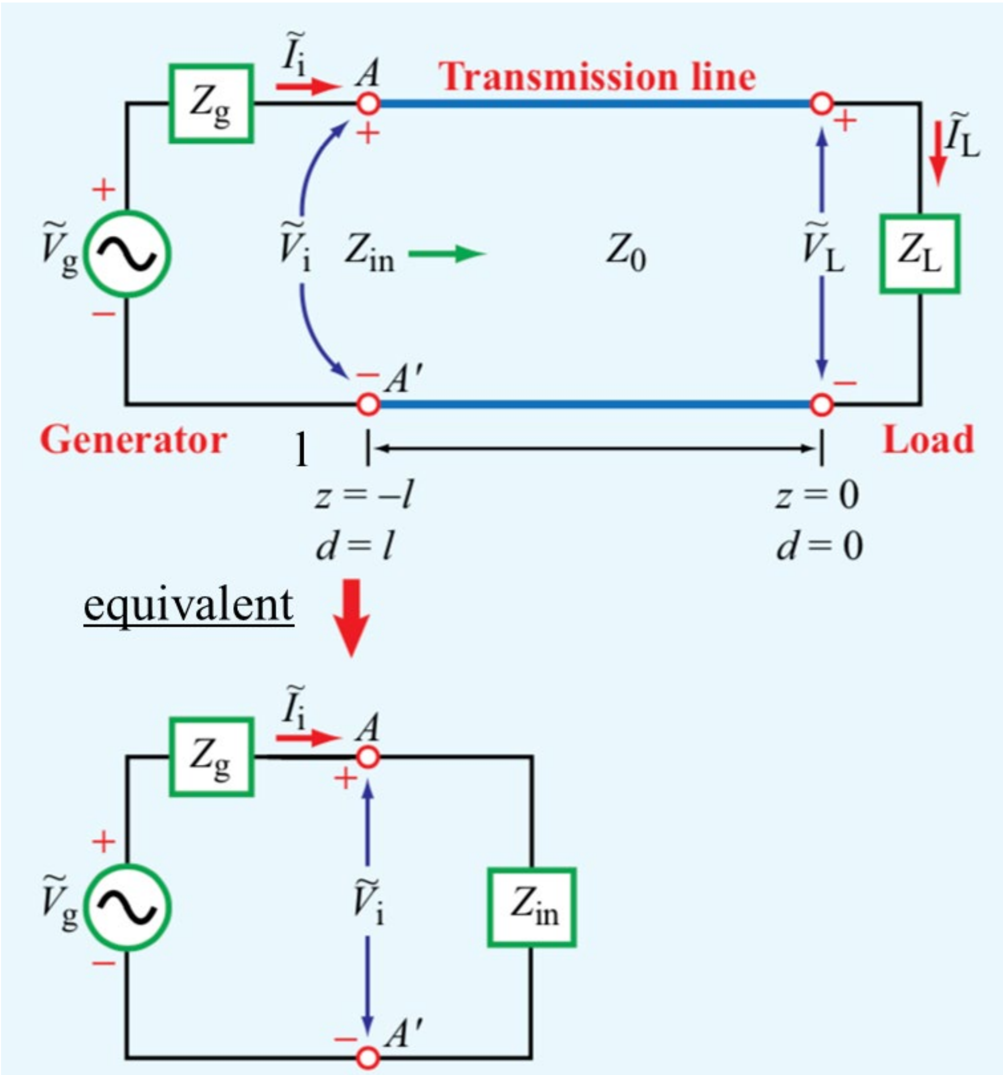
Using relations: $e^{j\beta l} = \cos(\beta l) + j sin(\beta \mathrm{l})$

$$e^{-j\beta l} = \cos(\beta l) - j sin(\beta \mathrm{l})$$

We obtain:

$$Z_{in} = Z_0\left(\frac{z_L\cos(\beta l) + j sin(\beta \mathrm{l})}{\cos(\beta l) + j z_L sin(\beta \mathrm{l})}\right) = Z_0\left(\frac{z_L + j tan(\beta \mathrm{l})}{1 + j z_L tan(\beta \mathrm{l})}\right)$$

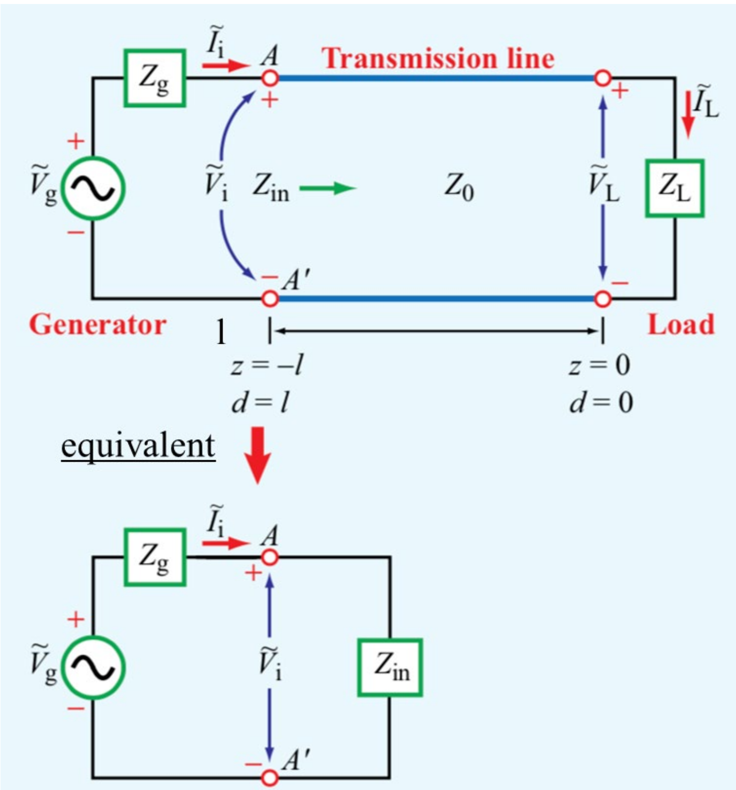
Wave Impedance – Lossless Line



At generator (source) end:

replace transmission line + load with input impedance = Z_{in}

Wave Impedance – Lossless Line



From generator: (transmission line replaced with impedance Z_{in})

$$\tilde{V}_i = \tilde{I}_i Z_{in} = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}$$

From transmission line:
(at input to line) → voltage:

$$\tilde{V}_i = \tilde{V}_i$$

Phasor voltage across Z_{in} equivalent at interface

$$\tilde{V}_i = \tilde{V}(-l) = V_0{}^{+}(e^{j\beta l} + \Gamma e^{-j\beta l})$$

$(at\ z = -l)$

Solve for $V_0{}^{+}$:

$$V_0{}^{+} = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$

$$\mathrm{Z(d)} = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^{+}(e^{j\beta d} + \Gamma e^{-j\beta d})}{V_0^{+}(e^{j\beta d} - \Gamma e^{-j\beta d})} Z_0$$

Transmission Line Recap

$$\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0 \qquad \frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0$$

Wave solution equations = traveling wave

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

4 unknowns: $V_0^+, V_0^-, I_0^+, I_0^-$

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-}$$

Reduced to 2 unknowns by transmission line lumped element model

Transmission Line Recap

Applying boundary condition at load, we can relate V_0^- to V_0^+ through Γ

Applying boundary condition at source, → obtain expression for V_0^+

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$

<u>From Load:</u> $\mathrm{Z(d)} = Z_0\left(\frac{1+\Gamma_d}{1-\Gamma_d}\right)$ $\qquad$ $\mathrm{Z(d)} = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+(e^{j\beta d}+\Gamma e^{-j\beta d})}{V_0^+(e^{j\beta d}-\Gamma e^{-j\beta d})} Z_0$

where $\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j\theta_r - j2\beta d} = |\Gamma| e^{j(\theta_r - 2\beta d)}$

<u>From Source:</u> $\mathrm{Z}(l) = Z_0\left(\frac{1+\Gamma_l}{1-\Gamma_l}\right)$ $\qquad$ $\Gamma_l = \Gamma e^{-j2\beta l} = |\Gamma| e^{j(\theta_r - 2\beta l)}$

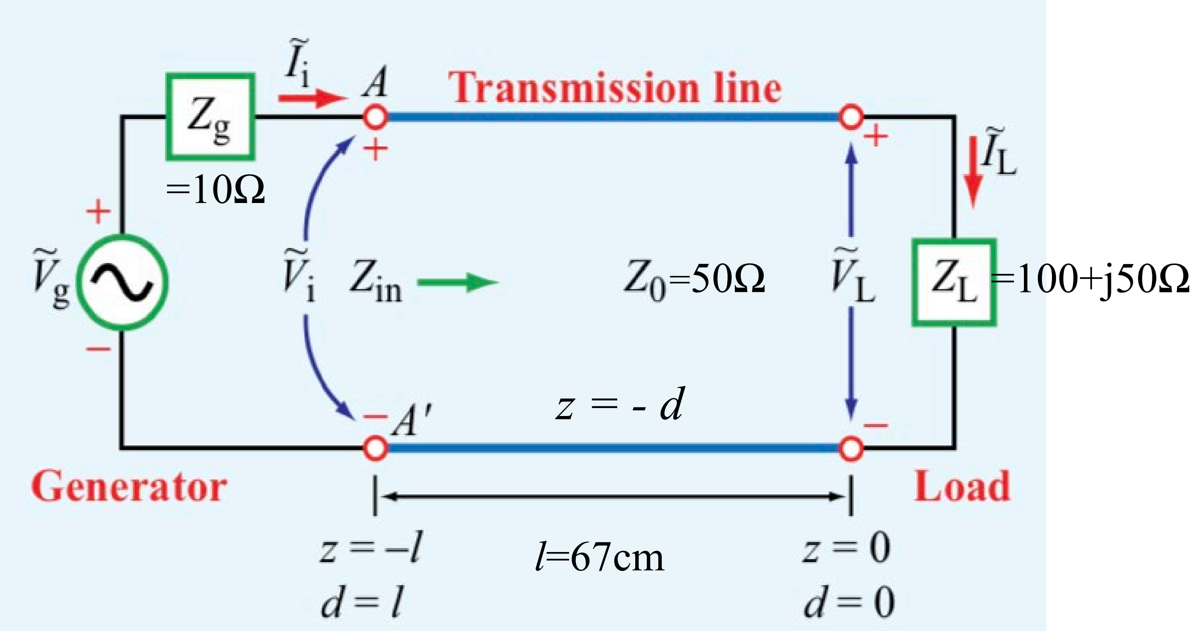
$$Z_{in} = Z(l = d) = Z_0\left(\frac{z_L + j\tan(\beta \mathrm{l})}{1 + j z_L \tan(\beta \mathrm{l})}\right)$$

$$\tilde{V}_i = \tilde{I}_i Z_{in} = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} = V_0^+(e^{j\beta l}+\Gamma e^{-j\beta l})$$

Example – solutions for *v(d,t)* and *i(d,t)*

Example:
Generator (f = 1.05GHz) circuit with $v_g(t) = 10\sin(\omega t + 30^o)$

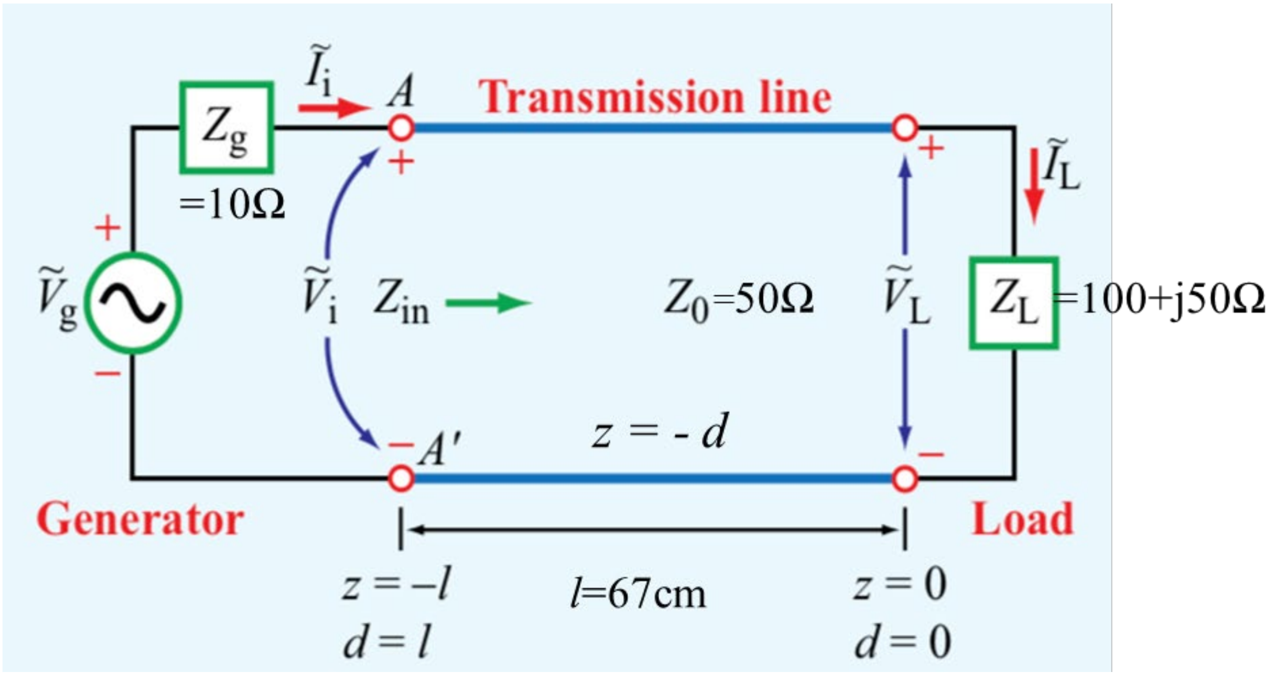
Connected to load: $Z_L = (100+j50)\ \Omega$
through lossless transmission line of length l = 67 cm and $Z_0 = 50\ \Omega$
$Z_g = 10\ \Omega$ and the phase velocity of line $u_p = 0.7c$



Find: v(d,t) and i(d,t)

Example – solutions for *v(d,t)* and *i(d,t)*

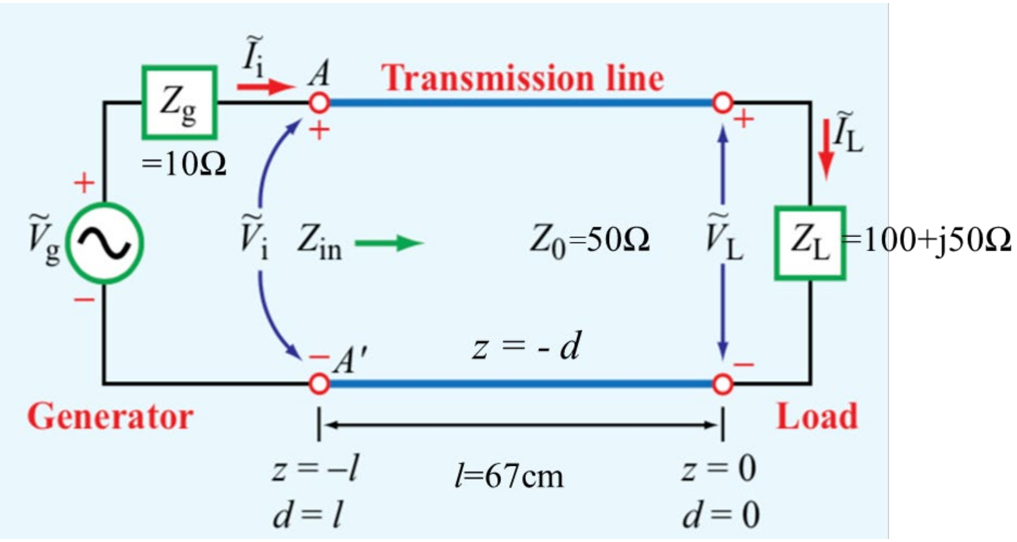
<u>Find</u>: v(d,t) and i(d,t)



Strategy: we will find $\tilde{V}(d)$ → then get time domain, *v(d,t)*

Example – solutions for *v(d,t)* and *i(d,t)*

Find: v(d,t) and i(d,t)



Is our TL lossless?

$$\text{To get } \tilde{V}(d) = \underbrace{V_0^+(e^{j\beta d}+\Gamma e^{-j\beta d})}_{\text{Lossless line solution}}$$

Need to find $V_0{}^+$:

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$

Need to obtain, $\tilde{V}_g$, Z_{in}, Γ, β

Example – solutions for *v(d,t)* and *i(d,t)*

Starting with getting β: $\beta = \dfrac{2\pi}{\lambda}$ $\quad u_p = \lambda f$

$$\lambda = \frac{u_p}{f} = \frac{0.7 \times 3 \times 10^8}{1.05 \times 10^9} = 0.2m$$

$$\beta = \frac{2\pi}{0.2} = 10\pi$$

$$\beta l = \frac{2\pi}{\lambda} l = \frac{2\pi}{0.2}(0.67) = 6.7\pi$$

$$\beta l = 6.7\pi - 6\pi = 0.7\pi \rightarrow 126^\circ$$

Example – solutions for *v(d,t)* and *i(d,t)*

Now obtain, Γ

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(100 + j50) - 50}{(100 + j50) + 50} = 0.45e^{j26.6^\circ}$$

$$Z_{in} = \mathrm{Z}(l)$$

$$= Z_0\left(\frac{1 + \Gamma_l}{1 - \Gamma_l}\right) = Z_0\left(\frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}}\right) \qquad \Gamma_l = \Gamma e^{-j2\beta l}$$

$$= 50\left(\frac{1 + 0.45e^{j26.6^\circ}\ e^{-j252^\circ}}{1 - 0.45e^{j26.6^\circ}\ e^{-j252^\circ}}\right)$$

$$\boxed{Z_{in} = (21.9 + j17.4)\ \Omega}$$

Example – solutions for *v(d,t)* and *i(d,t)*

Now obtain, $\widetilde{V_g}$ $\quad v_g(t) = 10\sin(\omega t + 30^o)$

$$v_g(t) = Re\left[\widetilde{V}_g e^{j\omega t}\right]$$

Need to move to
cosine reference

$$v_g(t) = 10\cos(90^\circ - \omega t - 30^o) = 10\cos(\omega t - 60^o)$$

$$v_g(t) = Re\left[10e^{-j60^\circ} e^{j\omega t}\right] \qquad \widetilde{V}_g = 10\angle -60\,^\circ$$

Example – solutions for *v(d,t)* and *i(d,t)*

Now, obtain $V_0{}^{+}$:

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$

$$V_0^+ = \left(\frac{10e^{-j60^\circ}(21.9 + j17.4)}{10 + (21.9 + j17.4)}\right)\left(\frac{1}{e^{j126^\circ} + 0.45e^{j26.6^\circ}\, e^{-j126^\circ}}\right)$$

$$V_0^+ = 10.2e^{j159^\circ}\ (V)$$

$$\tilde{V}(d) = V_0^+(e^{j\beta d} + \Gamma e^{-j\beta d})$$

$$\tilde{V}(d) = V_0^+(e^{j\beta d} + \Gamma e^{-j\beta d}) = 10.2e^{j159^\circ}\ (e^{j\beta d} + 0.45e^{j26.6^\circ}\ e^{-j\beta d})$$

Example – solutions for *v(d,t)* and *i(d,t)*

Now convert to time domain:

$$v(d,t) = Re\left[\tilde{V}(d)e^{j\omega t}\right]$$

$$\boxed{v(d,t) = 10.2\cos(\omega t + \beta d + 159°) + 4.55\cos(\omega t - \beta d + 185.6°)}$$

We can then obtain $\tilde{I}(d)$ and $\mathrm{i}(d,t)$:

$$\tilde{I}(d) = \frac{V_0^+}{Z_0}\left(e^{j\beta d} - \Gamma e^{-j\beta d}\right)$$

$$\tilde{I}(d) = 0.20e^{j159^\circ}\left(e^{j\beta d} - 0.45e^{j26.6^\circ}e^{-j\beta d}\right)$$

$$\boxed{\mathrm{i}(d,t) = 0.20\cos(\omega t + \beta d + 159°) - 0.09\cos(\omega t - \beta d + 185.6°)}$$